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Vortices and gauge fields in the pseudospin model of liquid ^{4}He

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Abstract. We show that below the lambda point a microscopic (pseudospin) model of liquid ^{4}He, that incorporates hard-core repulsion and nearest-neighbour attraction between atoms, is equivalent to a Gross–Pitaevskii (GP) model of weakly interacting (soft-core) bosons in the presence of a self-generated gauge field, by specializing to cylindrically symmetric vortex solutions. Requiring consistency in the behaviour of the condensate density in the two models, we are able to identify the gauge field in the GP model as that part of the total velocity that has a non-vanishing curl in the pseudospin model. It becomes evident that the gauge field is the depletion velocity.

1. Introduction

In order to provide a realistic description of the hydrodynamic properties of liquid ^{4}He below the lambda temperature T_λ, there have been several attempts to develop a gauge-theoretic formulation of superfluidity. Superfluid hydrodynamics has previously been studied by considering a model of weakly interacting bosons (Ginzburg and Pitaevskii 1958, Gross 1958, 1966), the interaction being a soft-core repulsion. This model (hereafter referred to as the GP model) leads to quantized vortices. However, there are two unphysical features associated with the model. First, the model predicts a continuity equation for the condensate density, totally ignoring the depletion effects due to interactions. Experimental studies (Sears *et al* 1982) indicate that the depletion effects are important even at very low temperatures (Olinto 1987). Second, the bulk velocity of the fluid is given in the GP model by $v_{GP} = (\hbar/m)\nabla\phi$ (ϕ being the phase of the order parameter and m the mass of a ^{4}He atom) leading to the vanishing of the vorticity, i.e. curl $v_{GP} = 0$ almost everywhere. However, at the exact centre of the vortex it is singular. For a physically realistic vortex, one expects curl v to be non-vanishing in a finite region within the vortex core (Gross 1963).

Work carried out by several authors (Sarfatt 1967, Thouless 1969, Cummings *et al* 1970, Chela-Flores 1977, Kawasaki and Brand 1985) shows that some aspects of the depletion can be incorporated by requiring the local gauge invariance of the Lagrangian in the GP theory. As may be expected, this requirement leads to coupled equations for the gauge and matter fields. It turns out that the gauge field appears as a depletion term in the continuity equation for the condensate density. However, in all these attempts, the gauge field has been treated like an externally imposed field, analogous to the discussion of the Meissner effect in superconductivity. But there is an important difference in the case of superfluidity, which must be stressed: any model

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that takes a realistic interaction between the ^{4}He atoms into account should *automatically* lead to such depletion effects. Therefore, the gauge field in the present context should be a *self-generated* one, rather than an externally imposed one. Furthermore, this makes it necessary to formulate the problem in such a way that the origin and physical significance of the gauge field become completely clear. This gives us ample motivation for comparing the results of a 'gauged' GP theory with those obtained in the framework of a model that incorporates the interaction more accurately than merely assuming a weak repulsion between atoms. In particular, we will show that the gauge field arises naturally within the framework of a realistic model.

The microscopic pseudospin model of superfluid ^{4}He (Matsubara and Matsuda 1956, Whitlock and Zilsel 1963) is well suited for our purposes. In this model, the presence of the hard-core interaction between ^{4}He atoms is taken into account by *postulating fermion-like behaviour for the matter field operators at the same site*, and boson-like behaviour for those at different sites. In addition, there is a nearest-neighbour (NN) attraction between atoms in this lattice model. Passing to the continuum limit of the model we find that the condensate density ρ_s is equal to the fluid density ρ minus the depletion term. We have recently derived a non-linear evolution equation for the order parameter and discussed various aspects of the model including the study of cylindrically symmetric vortex solutions (Balakrishnan *et al* 1989). The continuity equation for the *total* density in this theory leads to a vorticity that is non-vanishing within a vortex core of finite range. This feature essentially arises because of the appropriate inclusion of the repulsive interaction between the atoms. Finally, in this model a depletion term is obtained naturally in the evolution equation of the condensate density, making it possible to understand the physical origin of the gauge field.

The layout of this paper is as follows. In section 2, the 'gauged' GP model and the associated continuity equation for the condensate density ρ_s are presented. In section 3, the pseudospin model is briefly described and the corresponding continuity equation for the total density ρ is derived. In section 4, the equation of motion for ρ_s in the pseudospin model is derived, and the gauge field $\boldsymbol{A}$ (sometimes referred to as the 'depletion velocity field' in the literature) is identified by comparing the equation with the continuity equation for ρ_s obtained in section 2. Using this expression for $\boldsymbol{A}$, the solution for the condensate density $\rho_s(r)$ near the core centre and far away from it are determined for appropriate boundary conditions. The behaviour of this solution is shown to agree with the solution for $\rho_s(r)$ obtained by us from the equation of motion (Balakrishnan *et al* 1989) in an earlier paper. Our conclusions are discussed in section 5.

2. The gauged Gross–Pitaevskii model

As stated in the introduction, the hydrodynamics of superfluid ^{4}He has been studied using the theory of a weakly interacting bose system (Ginzburg and Pitaevskii 1958, Gross 1958, 1963). This GP model yields the following non-linear evolution equation for the complex order parameter or the condensate wavefunction ψ, defined as the expectation value of the annihilation operator for the bose field:

$$\mathrm{i}\hbar\partial\psi/\partial t + (\hbar^2/2m)\nabla^2\psi + V|\psi|^2\psi - \mu\psi = 0 \qquad (2.1)$$

on using the Hartree approximation. Here m is the mass of a ^{4}He atom, and μ is the chemical potential. The soft-core interaction energy $V(r)$ between two ^{4}He atoms is written in terms of the interaction strength V by defining $V(r) = V\Omega\delta(r)$, Ω being the volume of the system. Setting $\psi = \rho_s^{1/2}\exp(\mathrm{i}\phi)$ leads to the continuity equation

$$\partial\rho_s/\partial \mathrm{t} + (\hbar/m)\nabla\cdot(\rho_s\nabla\phi) = 0. \tag{2.2}$$

It is to be noted that $\rho_s = |\psi|^2$ is a dimensionless quantity and represents the probability density of the condensate fraction.

Now,

$$\mathrm{curl}\, \boldsymbol{v}_{\mathrm{GP}} = (\hbar/m)\mathrm{curl}\,\nabla\phi \equiv 0$$

in this model, and there is no depletion (i.e. $\rho = \rho_s$). However, it is well known that depletion effects due to interactions exist even at $T = 0$. Therefore it becomes necessary to improve this model. Among the various attempts in this direction, we shall briefly outline the approach of Cummings *et al* (1970), which is based on a suggestion of Sarfatt (1967). These authors assumed a Hamiltonian density which includes two types of 'probes'—a scalar field coupled to the mass density and a vector field coupled to the current density. Going beyond the Hartree–Fock model and using the concept of off-diagonal long-range order (Yang 1962), they obtained an equation for the order parameter. This equation contains a 'depletion velocity field' (with a non-vanishing curl) and a 'depletion chemical potential'. These fields are analogous to the vector and scalar potentials of conventional electrodynamics. Following these ideas Chela-Flores (1977) studied the effects introduced by imposing *local* gauge invariance on the GP Lagrangian:

$$L = -(\mathrm{i}\hbar/2)(\psi\partial_t\psi^* - \psi^*\partial_t\psi) - (\hbar^2/2m)\nabla\psi^*\cdot\nabla\psi + V|\psi|^4 - \mu|\psi|^2. \tag{2.3}$$

The modified gauge-invariant Lagrangian is found by postulating that the gauge field $\boldsymbol{A}$ is a local function, independent of time. In this case ∇ is replaced by the covariant derivative $(\nabla - \mathrm{i}(m/\hbar)\boldsymbol{A})$ in equation (2.3) and, in addition, a pure gauge field term $-\frac{1}{2}mL^2(\nabla\times\boldsymbol{A})^2$ appears in the Lagrangian. $\boldsymbol{A}$ is taken to have the dimensions of velocity. Consequently, dimensional requirements necessitate the introduction of a length scale L in this term. The coupled equations of motion for the matter and gauge fields are then given by

$$\mathrm{i}\hbar\partial\psi/\partial t = (-\hbar^2/2m)(\nabla - \mathrm{i}(m/\hbar)\boldsymbol{A})^2\psi - V|\psi|^2\psi + \mu\psi \tag{2.4}$$

and

$$L^2\nabla\times(\nabla\times\boldsymbol{A}) = (\hbar/2\mathrm{i}m)(\psi\nabla\psi^* - \psi^*\nabla\psi) - |\psi|^2\boldsymbol{A} = \rho_s[(\hbar/m)\nabla\phi - \boldsymbol{A}]. \tag{2.5}$$

Equation (2.4) leads to the following equation of continuity for $\rho_s = |\psi|^2$:

$$\partial\rho_s/\partial t + \nabla\cdot[\rho_s((\hbar/m)\nabla\phi - \boldsymbol{A})] = 0. \tag{2.6}$$

Rewriting equation (2.6) as

$$\partial\rho_s/\partial t + (\hbar/m)\nabla\cdot(\rho_s\nabla\phi) = \nabla\cdot(\rho_s\boldsymbol{A}) \tag{2.7}$$

we see that the introduction of the gauge field has provided the (physically required) depletion term in the continuity equation for the condensate density.

To proceed, one must solve the coupled set of equations (2.4) and (2.5). As is clear from the manner in which the gauge field $\boldsymbol{A}$ has been introduced in the foregoing, the earlier works have considered the gauge field to be purely external. However, in the present case, $\boldsymbol{A}$ is introduced in order to describe intrinsic 'frictional' effects in the system, arising ultimately from the interaction between atoms. Thus $\boldsymbol{A}$ *must* depend implicitly on the interatomic interaction. But this same interaction determines the matter field ψ as well. Hence, in a consistent theory, $\boldsymbol{A}$ must be determined as a *self-generated* gauge field which depends on the matter field. It is evident that for this purpose we need a model that incorporates the interactions in a reasonably realistic manner. The GP model is inadequate in this regard. In the next section we turn to an improved (pseudospin) model, with the ultimate aim of identifying the appropriate gauge field $\boldsymbol{A}$ from the formalism itself, instead of introducing it as an external field.

3. The pseudospin model

The pseudospin model (Matsubara and Matsuda 1956) is deduced by starting with a system of hard-core bosons with a (NN) attractive interaction. The hard core is incorporated by demanding fermion-like anticommutation relations for the field operators at the same site, while boson-like commutation relations hold for operators belonging to different sites. The field operators behave like $S = \frac{1}{2}$ spin-flip operators, and the system is represented by an anisotropic Heisenberg pseudospin Hamiltonian:

$$H = -\sum_l \left\{ (b-\mu)S_l^z + \sum_\delta (\hbar^2/4ma^2) \left[\sum_{\alpha=x,y} S_l^\alpha S_{l+\delta}^\alpha + v_0 S_l^z S_{l+\delta}^z \right] \right\} \tag{3.1}$$

where $b = 3[(\hbar^2/ma^2) - v_0]$ and $-v_0 (v_0 > 0)$ is the NN attractive interaction between two helium atoms separated by the lattice spacing a. In (3.1), l refers to the site index and the summation over δ indicates a summation over NN pairs. Using this pseudospin model, Whitlock and Zilsel (1963) have studied the thermodynamic properties and the nature of the elementary excitations in the superfluid.

In an earlier paper (Balakrishnan *et al* 1989, hereafter referred to as I) we have analysed this model using spin coherent states to understand the non-linear hydrodynamics of superfluid ^{4}He. In this formalism the superfluid order parameter is specified by $\eta_l = \langle S_l^+ \rangle$, the spin coherent state average. This quantity may be represented as

$$\eta_l \equiv \langle S_l^+ \rangle = \tfrac{1}{2} \sin\theta_l \exp(\mathrm{i}\phi_l) \tag{3.2}$$

where θ_l and ϕ_l stand for the polar and azimuthal angles of the classical spin vector at site l. Also,

$$\langle S_l^z \rangle = \tfrac{1}{2}\cos\theta_l = \tfrac{1}{2}(1 - 4|\eta_l|^2)^{1/2} \tag{3.3}$$

and

$$\langle \rho_l \rangle = \tfrac{1}{2} - \langle S_l^z \rangle = \sin^2(\theta_l/2). \tag{3.4}$$

It is evident that η_l and ρ_l are related by

$$|\eta_l|^2 = \rho_l(1-\rho_l) \tag{3.5}$$

and that $|\eta_l|^2 \leqslant \frac{1}{4}$.

Using the equation of motion

$$\mathrm{i}\hbar\partial S_l^+/\partial t + [H, S_l^+] = 0 \tag{3.6}$$

and passing to the continuum version, we find (see I for details) the following evolution equation for η:

$$\begin{aligned}\mathrm{i}\hbar\frac{\partial\eta}{\partial t} = \{b[1-(1-4|\eta|^2)^{1/2}] - \mu\}\eta - (\hbar^2/2m)(1-4|\eta|^2)^{1/2}\nabla^2\eta \\ - v_0 a^2\eta\{\nabla^2|\eta|^2 + 2(\nabla|\eta|^2)^2/(1-4|\eta|^2)\}/(1-4|\eta|^2)^{1/2}.\end{aligned} \tag{3.7}$$

We note that the continuum version of equation (3.5) is

$$|\eta|^2 = \rho_{\mathrm{s}} = \rho(1-\rho). \tag{3.8}$$

In other words the condensate density ρ_{s} is equal to the fluid density ρ minus the 'depletion' ρ^2 resulting essentially from the hard-core repulsion between atoms. It is easily seen that the GP equation (2.1) is obtained from equation (3.7) in the leading small amplitude approximation ($|\eta|^2 \ll \frac{1}{4}$), on identifying $2b$ with v_0. Similarly, by retaining non-linear terms in a suitable manner one can recover the phenomenological equation of Rutledge *et al* (1978) for ^{4}He films. Full linearization of equation (3.7) yields the Bogoliubov spectrum for the elementary excitations of the superfluid (see I).

By setting $\eta = [\rho(1-\rho)]^{1/2}\exp(\mathrm{i}\phi) = \rho_{\mathrm{s}}^{1/2}\exp(\mathrm{i}\phi)$ in equation (3.7) we obtain the following continuity equation:

$$\partial\rho/\partial t + (\hbar/m)\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0. \tag{3.9}$$

Thus we see that the total velocity $\boldsymbol{v}$ of the liquid may be identified as

$$\boldsymbol{v} = (\hbar/m)(1-\rho)\boldsymbol{\nabla}\phi. \tag{3.10}$$

It is clear that, because of the presence of the depletion term, curl $\boldsymbol{v}$ is not identically equal to zero.

We now derive an evolution equation for a complex function χ defined by $\chi = \rho^{1/2}\exp(\mathrm{i}\phi)$ where ρ is the *total* density, just as ρ_{s} is the condensate density. The new set of variables χ_l and χ_l^* is obtained from the spin coherent state averaged variables as

$$\chi_l = \rho_l^{1/2}\exp(\mathrm{i}\phi_l) = \sin(\theta_l/2)\exp(\mathrm{i}\phi_l). \tag{3.11}$$

The expectation values of the spin operators are given in terms of χ_l and χ_l^* by

$$\begin{aligned}\langle S_l^x\rangle &= [\rho_l(1-\rho_l)]^{1/2}\cos\phi_l = (1-\rho_l)^{1/2}(\chi_l+\chi_l^*)/2 \\ \langle S_l^y\rangle &= (1-\rho_l)^{1/2}(\chi_l-\chi_l^*)/2\mathrm{i}\end{aligned} \tag{3.12}$$

and

$$\langle S_l^z\rangle = \tfrac{1}{2} - |\chi_l|^2$$

where we have made use of equations (3.2) to (3.5). The amplitudes are related as follows:

$$|\eta_l|^2 = |\chi_l|^2(1 - |\chi_l|^2). \tag{3.13}$$

The expectation value of the Hamiltonian density H can be written as

$$\begin{aligned}\langle H_l\rangle = &-(b-\mu)(\tfrac{1}{2}-\rho_l) + (\hbar^2/8ma^2)\{(1-\rho_l)^{1/2}(\chi_l+\chi_l^*)[(1-\rho_{l+1})^{1/2}(\chi_{l+1}+\chi_{l+1}^*) \\ &+ (1-\rho_{l-1})^{1/2}(\chi_{l-1}+\chi_{l-1}^*)] + (1-\rho_l)^{1/2}(\chi_l^*-\chi_l)[(1-\rho_{l+1})^{1/2} \\ &\times(\chi_{l+1}^*-\chi_{l+1}) + (1-\rho_{l-1})^{1/2}(\chi_{l-1}^*-\chi_{l-1})]\} \\ &+ v_0\{(\tfrac{1}{2}-\rho_l)[1-(\rho_{l+1}+\rho_{l-1})]\}.\end{aligned} \tag{3.14}$$

For a classical spin system, the canonically conjugate variables are S_l^z and ϕ_l with the Poisson bracket relation $[\phi_l, S_{l'}^z] = \delta_{l,l'}$. Using the relationship in equation (3.4), ϕ_l and $-\rho_l$ may be treated as the conjugate variables, leading to the following Lagrangian density in the continuum limit (with $\mathcal{H}$ denoting $\langle H\rangle$ in this limit)

$$\begin{aligned}\mathcal{L} = -\rho(r,t)\dot{\phi}(r,t) - \mathcal{H} = &\{[\chi^*(\partial\chi/\partial t) - \chi(\partial\chi^*/\partial t)]/2\} \\ &- \mathcal{H}(\chi,\chi^*,\nabla\chi,\nabla\chi^*,\nabla^2\chi,\nabla^2\chi^*).\end{aligned} \tag{3.15}$$

The second-order derivatives in this equation for $\mathcal{L}$ may be eliminated by using integration by parts, and hence

$$\begin{aligned}\mathcal{L} = &(\mathrm{i}/2)(\chi^*(\partial\chi/\partial t) - \chi(\mathrm{d}\chi^*/\partial t)) + (b-\mu)(\tfrac{1}{2}-|\chi|^2) + (\hbar^2/ma^2)|\chi|^2(1-|\chi|^2) \\ &+ \tfrac{1}{2}v_0(1-2|\chi|^2) - (\hbar^2/2m)(1-|\chi|^2)(\nabla\chi^*\cdot\nabla\chi) \\ &+ ((\hbar^2/4m) - v_0a^2)(\chi^*\nabla\chi - \chi\nabla\chi^*)^2.\end{aligned} \tag{3.16}$$

The Euler–Lagrange equation corresponding to equation (3.16) is

$$\begin{aligned}\mathrm{i}\frac{\partial\chi}{\partial t} - &\left[(b-\mu) - (1-2|\chi|^2)\left(\left(\frac{\hbar^2}{ma^2}\right) - 2v_0\right)\right]\chi \\ &+ \left(\frac{\hbar^2}{2m}\right)(\nabla\chi\cdot\nabla\chi^*)\chi - \left(\frac{\hbar^2}{2m}\right)\nabla|\chi|^2\cdot\nabla\chi \\ &+ \left(\frac{\hbar^2}{2m}\right)(1-|\chi|^2)\nabla^2\chi - 2\left(\left(\frac{\hbar^2}{4m}\right) - a^2v_0\right)(\nabla^2|\chi|^2)\chi = 0.\end{aligned} \tag{3.17}$$

It is interesting to note that the structure of this evolution equation for χ (which determines the total density ρ) is quite different from that of equation (3.7) for η (which determines $\rho_s = |\eta|^2$, the density of the condensate). As a check on this Lagrangian approach, we mention that equation (3.9) can be recovered by an explicit evaluation of $\partial\rho/\partial t = \chi^*(\partial\chi/\partial t) + \chi(\partial\chi^*/\partial t)$ from equation (3.17).

4. The gauge field A

We begin by summarizing certain relevant aspects of the microscopic pseudospin model discussed in I. These will subsequently be used in establishing the connection between the rotational part of the total velocity in this model and the gauge field $\boldsymbol{A}$. It has been shown in I that equation (3.7) for the complex order parameter η in this model supports vortex solutions. In particular for a vortex with winding number $n = 1$ the condensate density $\rho_s(r)$ has the following behaviour (see equation (5.14) of I):

$$\lim_{r\to 0}\rho_s(r) \simeq (r/\xi)^2 \qquad \lim_{r\to\infty}\rho_s(r) \simeq \text{constant}. \tag{4.1}$$

The core size ξ has been estimated as

$$\xi = \hbar/(4mbf_0^2)^{1/2}$$

where $b = 3[(\hbar^2/ma^2) - v_0]$ and f_0^2 denotes the equilibrium condensate density.

We proceed by first observing that equation (3.7) leads to the following *exact* equation for $\rho_s = |\eta|^2$:

$$\frac{\partial\rho_s}{\partial t} + \frac{\hbar}{m}(1-4\rho_s)^{1/2}\nabla\cdot[\rho_s\nabla\phi] = 0. \tag{4.2}$$

For vortex solutions we have (by definition)

$$\nabla\rho_s\cdot\nabla\phi = 0. \tag{4.3}$$

Therefore equation (4.2) can be written as

$$\frac{\partial\rho_s}{\partial t} + \frac{\hbar}{m}\nabla\cdot(\rho_s\nabla\phi) = \frac{\hbar}{m}\nabla\cdot([1-(1-4\rho_s)^{1/2}]\rho_s\nabla\phi). \tag{4.4}$$

Now let us consider the gauged GP model where the superfluid is described by the two coupled equations (2.4) and (2.5). In section 2 it was shown that the equation for the matter field, (2.4), leads to equation (2.7) for $(\partial\rho_s/\partial t)$. Comparing this equation with (4.4) we obtain

$$\frac{\hbar}{m}\nabla\cdot\{[1-(1-4\rho_s)^{1/2}]\rho_s\nabla\phi\} = \nabla\cdot(\boldsymbol{A}\rho_s). \tag{4.5}$$

This identifies $\boldsymbol{A}$ as

$$\boldsymbol{A} = (\hbar/m)[1-(1-4\rho_s)^{1/2}]\nabla\phi + (1/\rho_s)\,\text{curl}\,\boldsymbol{\lambda} = (2\hbar/m)\rho\nabla\phi + (1/\rho_s)\,\text{curl}\,\boldsymbol{\lambda} \tag{4.6}$$

where we have used the relationship (see equation (3.8))

$$\rho = \tfrac{1}{2}[1-(1-4\rho_s)^{1/2}] \tag{4.7}$$

and $\boldsymbol{\lambda}$ is an arbitrary vector. However, the choice of $\boldsymbol{\lambda}$ should be such that when this expression for $\boldsymbol{A}$ is substituted into the equation for the *gauge field*, i.e. equation (2.5), one must obtain a solution for $\rho_s(r)$ whose behaviour is *consistent* with the behaviour (equation (4.1)) displayed by the microscopic model.

We are guided by the following physical considerations: $\boldsymbol{A}$ should be a 'self-generated' gauge field as discussed in the introduction. Therefore curl $\boldsymbol{\lambda}$ must be a function of ρ_s and ϕ which describe the condensate. Now, in the microscopic model, we have shown that $\boldsymbol{v} = (\hbar/m)(1-\rho)\boldsymbol{\nabla}\phi$. This suggests that we write the expression for $\boldsymbol{v}$ as

$$\boldsymbol{v} = (\hbar/m)\boldsymbol{\nabla}\phi - \boldsymbol{A}. \tag{4.8}$$

It then becomes evident that the second term

$$\boldsymbol{A} = (\hbar/m)\rho\boldsymbol{\nabla}\phi \equiv (\hbar/2m)[1-(1-4\rho_s)^{1/2}]\boldsymbol{\nabla}\phi \tag{4.9}$$

is indeed responsible for making curl $\boldsymbol{v} \neq 0$, i.e. it causes depletion effects. Equations (4.9) and (4.6) together yield

$$\mathrm{curl}\,\boldsymbol{\lambda} = (-\hbar/m)\rho_s\rho\boldsymbol{\nabla}\phi.$$

At this point it is appropriate to mention the work of Kawasaki and Brand (1985). They have 'defined' a certain expression for $\boldsymbol{v}$ (different from ours) which has a non-zero curl by construction, to take depletion into account. The advantage of our approach is that an expression for $\boldsymbol{v}$ with this required property has been derived starting from first principles.

It remains to substitute the expression for $\boldsymbol{A}$ given in equation (4.9) into equation (2.5) and show that the resulting differential equation displays the required behaviour (4.1) for $\rho_s(r)$. To do this, it is convenient to work in terms of a variable y:

$$y = (1-4\rho_s)^{1/2}. \tag{4.10}$$

Thus equation (4.9) becomes

$$\boldsymbol{A} = (\hbar/2m)(1-y)\boldsymbol{\nabla}\phi.$$

Specializing to the case of a single vortex solution with cylindrical symmetry we obtain

$$\begin{aligned} L^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) &= -\left(\frac{\hbar L^2}{2m}\right)\boldsymbol{\nabla}\times\boldsymbol{\nabla}y\times\boldsymbol{\nabla}\phi \\ &= \left(\frac{\hbar L^2}{2m}\right)[(\boldsymbol{\nabla}y\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}\phi(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}y)] \\ &= \left(\frac{\hbar L^2}{2m}\right)\frac{1}{r}\left[\frac{\mathrm{d}^2y}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}y}{\mathrm{d}r}\right]\hat{e}_\varphi. \end{aligned} \tag{4.11}$$

Also,

$$\rho_s\left[\frac{\hbar}{m}\boldsymbol{\nabla}\phi - \boldsymbol{A}\right] = \left(\frac{\hbar}{8m}\right)(1-y^2)(1+y)\frac{1}{r}\hat{e}_\varphi. \tag{4.12}$$

Hence using equations (4.11) and (4.12) in equation (2.5) we obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}y}{\mathrm{d}r} = \frac{1}{4L^2}(1-y^2)(1+y). \tag{4.13}$$

Replacing $(1-y)$ by R

$$-\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}=\frac{1}{4L^2}R(2-R)^2. \tag{4.14}$$

We will solve this non-linear equation in an iterative fashion as follows: making the substitution $R = Qr$ in the linearized version of this equation we obtain the following (Bessel) equation (Abramowitz and Stegun 1986)

$$r^2\left(\frac{d^2Q}{dr^2}\right)+r\left(\frac{dQ}{dr}\right)+Q\left(\left(\frac{r^2}{L^2}\right)-1\right)=0 \tag{4.15}$$

with the solution

$$Q = CJ_1(r/L)+DY_1(r/L). \tag{4.16}$$

Physical considerations demand that ρ_s should vanish at the centre of the core. Hence R should approach zero as $r \to 0$. However, $Y_1 \simeq L/r$ as $r \to 0$. Hence D is chosen to be zero. Since $J_1(r) \simeq r$ as $r \to 0$,

$$Q \simeq (r/L). \tag{4.17}$$

Consequently we may write (on using equation (4.10))

$$R \simeq (1-(1-4\rho_s)^{1/2}) \simeq Cr^2/L \tag{4.18}$$

where C should be proportional to $(L)^{-1}$, L being the length scale in the problem. Comparison with equation (4.1) points out that L should represent the core radius ξ, which in turn depends on the microscopic parameters as discussed earlier in this section. It may be noted that the next iteration gives a higher order term in R, which can once again be neglected as $r \to 0$. Hence the behaviour for small r remains the same as in equation (4.18).

We now discuss the behaviour at the other limit, i.e. $r \to \infty$. In this limit $y = (1-4\rho_s)^{1/2}$ should approach zero at low temperatures in the pseudospin model. Putting $y = \tilde{Q}r$ and linearizing equation (4.13) we obtain

$$x^2\frac{d^2\tilde{Q}}{dx^2}+x\frac{d\tilde{Q}}{dx}-\tilde{Q}(1+x^2)=\frac{x}{2L} \tag{4.19}$$

where $x = r/2L$. The homogeneous equation corresponding to equation (4.19) has solutions $I_1(x)$ and $K_1(x)$. To obtain the solutions for the inhomogeneous equation we have to construct the Green function $g(x,\xi)$ where the range of the variables is given by $0 \leqslant (x,\xi) \leqslant b_0$ (b_0 being large). Thus for equation (4.19) we have

$$\begin{aligned} g(x,\xi) &= -U_1(x)U_2(\xi)/P_0(\xi)W(U_1,U_2;\xi) && 0<\xi<x \\ &= U_2(x)U_1(\xi)/P_0(\xi)W(U_1,U_2;\xi) && x<\xi<b_0 \end{aligned} \tag{4.20}$$

with $P_0(\xi) = \xi^2$. U_1 and U_2 stand for the modified Bessel functions I_1 and K_1. W is the Wronskian

$$W(I_1,K_1;\xi) = -\xi^{-1}. \tag{4.21}$$

Thus the total solution is

$$U(x) = -U_1(x)\int_0^x \left[\frac{U_2(\xi)f(\xi)}{P_0(\xi)W(U_1,U_2;\xi)}\right] \mathrm{d}\xi \\ - U_2(x)\int_x^{b_0} \left[\frac{U_1(\xi)f(\xi)}{P_0(\xi)W(U_1,U_2;\xi)}\right] \mathrm{d}\xi + \tilde{C}_1 U_1(x) + \tilde{C}_2 U_2(x) \qquad (4.22)$$

with $f(\xi) = \xi/2L$.

When x assumes the value b_0 which eventually tends to large values, the solution is

$$U(b_0) = \left\{\tilde{C}_1 - \int_0^{b_0} \left[\frac{U_2(\xi)f(\xi)}{P_0(\xi)W(U_1,U_2;\xi)}\right] \mathrm{d}\xi\right\} U_1(b_0) + \tilde{C}_2 U_2(b_0) \qquad \text{as } b_0 \to \infty. \qquad (4.23)$$

It is to be pointed out that

$$[f(\xi)/P_0(\xi)W(U_1,U_2;\xi)] = -(2L)^{-1}. \qquad (4.24)$$

If C_1 *is chosen such that the curly bracket in equation* (4.23) is zero for $b_0 \to \infty$ we obtain

$$\tilde{Q} = U(x) \to \tilde{C}_2(\pi/2x)^{1/2}\exp(-x) \qquad x = r/2L.$$

Thus

$$y \simeq \tilde{C}_2(r\pi L)^{1/2}\exp(-r/2L) \to 0 \qquad \text{as } r \to \infty.$$

Hence ρ_s tends to its maximum value at large distances from the centre of the vortex, in agreement with equation (4.1).

Thus using the self-generated gauge field $\boldsymbol{A}$, the behaviour for ρ_s for both $r \to 0$ and $r \to \infty$ (for a single vortex solution) is seen to be consistent with the corresponding behaviour in the microscopic model.

5. Concluding remarks

We have shown that the pseudospin model of superfluid ^{4}He is equivalent to a gauged GP model with the gauge field $\boldsymbol{A}$ identified as the non-zero curl part of the total velocity, given by

$$\boldsymbol{A} = (\hbar/m)\rho\nabla\phi = (\hbar/2m)[1-(1-4\rho_s)^{1/2}]\nabla\phi \qquad (5.1)$$

provided $\nabla\rho_s \cdot \nabla\phi = 0$. This is definitely satisfied for the cylindrically symmetric vortex solutions considered by us, for which $\nabla^2\phi = 0$. Also, the relationship curl $\boldsymbol{\lambda} = (-\hbar/m)\rho\rho_s\nabla\phi$ (see below equation (4.9)) is automatically satisfied. Thus $\nabla \cdot \boldsymbol{A} = 0$ and equation (2.5) reduces to

$$\boldsymbol{A} - (L^2/\rho_s)\nabla^2\boldsymbol{A} = (\hbar/m)\nabla\phi. \qquad (5.2)$$

By using a phenomenological theory, Thouless (1969) has analysed the hydrodynamics of a dense superfluid and derived the following equation for $\boldsymbol{v}$, the total velocity below T_λ:

$$\boldsymbol{v} - \lambda_0^2 \nabla^2 \boldsymbol{v} = (\hbar/m)\nabla\phi \tag{5.3}$$

where ϕ is the phase of the condensate wavefunction and λ_0^2 is proportional to $(\rho_s)^{-1}$. In spite of the similarity between equations (5.2) and (5.3), we assert that $\boldsymbol{A} \neq \boldsymbol{v}$. In fact $\boldsymbol{v}$ is given by

$$\boldsymbol{v} = (\hbar/m)\nabla\phi - \boldsymbol{A} \tag{5.4}$$

which, as we have shown, is consistent with the vortex solution of $\rho_s(r)$. Equations (2.5) and (2.6) together imply that $(\partial\rho_s/\partial t) = 0$ for all time. The natural interpretation of the length scale L as characteristic of the variation in $\boldsymbol{A}$ emerges by comparison with the length scale of the vortex solutions given in I. Time-dependent solutions will be studied using time-dependent gauge fields in a future contribution.

If one computes the line integral of the gauge field in equation (5.1) over a circuit with a radius which is very large when compared with the vortex core radius we obtain

$$\oint_{\mathrm{R}} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = (n\hbar/m)k. \tag{5.5}$$

where

$$k = \tfrac{1}{2}[1 - (1 - 4\rho_s^0)^{1/2}]. \tag{5.6}$$

In this equation n denotes the winding number, and the temperature-dependent quantity ρ_s^0 denotes the constant value of ρ_s obtained at a large distance from the vortex core. Using Gauss's theorem, the strength of the monopole corresponding to equation (5.1) is (Dirac 1978)

$$\mu = (\hbar k/2m)n. \tag{5.7}$$

In conclusion, we have been able to identify the pseudospin model of superfluid ^{4}He with a gauged GP model by specializing to cylindrically symmetric vortex solutions of the condensate order parameter. Since our model incorporates the hard-core interaction between the bosons in an effective manner, it has the advantage of explicitly displaying the depletion characteristics present in a realistic system. This leads to the result that the gauge field is self-generated by the internal interactions in the system.

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